Random walks along the streets and canals in compact cities: Spectral analysis, dynamical modularity, information, and statistical mechanics

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Different models of random walks on the dual graphs of compact urban structures are considered. Analysis of access times between streets helps to detect the city modularity. The statistical mechanics approach to the ensembles of lazy random walkers is developed. The complexity of city modularity can be measured by an informationlike parameter which plays the role of an individual fingerprint of *Genius loci*. Global structural properties of a city can be characterized by the thermodynamic parameters calculated in the random walk problem.

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I. INTRODUCTION TO CITY NETWORK STUDIES

Studies of urban networks have a long history. Many researches have been devoted to the optimizations of transportation routes and power grids, to the predictions of traffic flows between the highly populated city districts, and to the investigations of habits and artifact exchanges between distanced settlements in historical eras. In most of them, relations between different components of urban structure are often measured along streets, the routes between junctions which form the nodes of an equivalent planar graph. The graph-theoretic principles have been applied in [1] in order to measure the hierarchy in regional central place systems and in [2] to the measurement of transportation networks. The use of graph-theoretic view and network analysis of spatial systems in geographic science is reviewed in [3]. It is interesting to mention that graphs have been widely used to represent the connectivity between offices in buildings [4] and to classify various building types in [5].

In all these studies, the traffic end points and junctions were treated as nodes and the routes were considered as edges of some planar graphs. Being embedded into the geographical and economical landscapes, these planar graphs bare their multiple fingerprints. Among the main factors featuring them are the high costs of the maintenance of long-range connections and the scarce availability of physical space. Spatial networks differ from other complex networks and call for alternative approaches to investigate them [6].

While studying the motifs and cycles in the complex networks, a comparative analysis of the original graph and of its randomized version is used. If a motif is statistically significant, it appears in the real network much more frequently than in randomized versions of the graph [7]. However, in planar city street patterns, its randomized version is not of significance since, first, it is surely a nonplanar graph due to the randomness of edge crossings and, second, the long-range connections which inevitably are present in random graphs in abundance are extremely costly in real cities [6]. In [8], it has been proposed to compare city street patterns with gridlike structures, which is indeed useful essentially for regular urban development.

It was formulated in the classical essay of Hughes [9] that the accessibility of important city objects for vehicle traffic and pedestrians is always the chief factor in regulating the growth and expansion of the city. A broad, simple scheme of main traffic lines gives a sense of connectedness and unity to the various parts of the city and links up country and town. In [10,11], a significant correlation between the topological accessibility of streets and their popularity, microcriminality, microeconomic vitality, and social livability had been established. In the traditional representation of space syntax based on relations between streets through their junctions, the accessibility or distance is associated with points or junctions.

We would like to mention that the issues of global connectivity of finite graphs and accessibility of their nodes are the classical fields of researches in graph theory. They are studied by means of certain dynamical processes defined on the graphs. In particular, in order to reach "obscure" parts of large sets and estimate the probable access times to them, *random walks* are often used [12]. There are a number of other processes that can be defined on a graph describing various types of diffusion of a large number of random walkers moving on the network at discrete time steps [13]. In all such processes we have deal with discrete time Markov chains studied in probability theory. Markov chains have the property that their time evolution behavior depends only upon their current state and the state transition properties of the model.

At each time step every walker moves from its current node to one of the neighboring nodes along a randomly selected link. The metric distance between the nodes is of no matter for such a discrete time diffusion process and then the focus of study is naturally shifted from the original problem of traditional space syntax to a *dual* one based on relations between the streets which themselves are treated as nodes. The distance between two streets, in such a representation, is a distance in the graph-theoretic sense. The dual graphs of a geographic network comparable in its structure with other complex networks are irrelevant to either distance or physical space constraints. The relations between the traditional space syntax representation based on the relations between streets through their junctions and the dual representation

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that is a morphological representation of relations between junctions through their streets have been studied in detail in [14]. Dual city graphs have been developed and studied within the concept of space syntax [10]. The key characteristics in space syntax is that precedence is given to linear features such as streets in contrast to fixed points which approximate locations [11]. Dual city graphs have been discussed recently in [15]. In [16], they are called the information city network.

In [17], while identifying a street over a plurality of routes on a city map, the "named-street" approach has been used, in which two different arcs of the original street network are assigned to the same street ID provided they have the same street name. The main problem of the approach is that the meaning of a street name could vary from one district or quarter to another even within the same city. For instance, the streets in Manhattan do not meet, in general, the continuity principle, rather playing the role of local geographical coordinates.

Being interested in the statistics of random walks, we generalized the approach used in [17] in a way to account the possible discontinuities of streets. Namely, we assign an individual street ID code to each continuous part of it even if all of them share the same street name. The dual graph is constructed by mapping edges coded with the same street ID into nodes of the dual graph and intersections among each pair of edges in the original graph into edges connecting the corresponding nodes of the dual graph as was done in [17].

In [15], an intersection continuity (ICN) principle different from our identification approach has been used: two edges forming the largest convex angle in a crossroad on the city map are assigned the highest continuity and therefore are coupled together, acquiring the same street ID. The main problem with the ICN principle is that the streets crossing under convex angles would artificially exchange their identifiers, which is not crucial for the study of degree statistics, but makes it difficult to interpret the results on random walks and detect the dynamical modularity of the city.

It is also important to mention that the number of street ID's identified within the ICN principle usually exceeds substantially the actual number of street names in a city. In [6,15,18–20], degree statistics and various centrality measures for the data set of square mile samples of different world cities investigated. However, the decision on which a square mile would provide an adequate representation of a city is always questionable.

In this paper, we use an alternative strategy investigating the spectral properties of random walks defined on the *dual* graphs of *compact* city patterns bounded by natural geographical limitations. The reason we consider compact urban domains is twofold. First, it allows us to avoid the problem of a "square mile" and, second, compact urban domains have been usually developed "at once," in accordance with certain architectural principles; their partial redevelopment has been occasional and rear, so that they can be considered typical.

We have studied Markov chains defined on six *undirected* dual graphs corresponding to the different street and canal urban structures. Two of them are situated on islands: Manhattan (with an almost regular gridlike city plan) and the network of Venice canals (imprinting the joined effect of natural, political, and economical factors acting on the network during many centuries). We have also considered two cities founded shortly after the Crusades and developed within medieval fortresses: Rothenburg ob der Tauber (o.d.T.) (the medieval Bavarian city preserving its original structure from the 13th century) and the Bielefeld downtown (Altstadt Bielefeld) composed of two different parts: the old one founded in the 13th century and the modern part subjected to partial urban redevelopment at the end of the 19th century. To supplement the study, we have investigated the canal network of the city of Amsterdam. Although it is not actually isolated from the national canal network, it is binding to the delta of the Amstel river, forming a dense canal web showing a high degree of radial symmetry. The scarcity of physical space is among the most important factors determining the structure of compact urban patterns. The general information on the dual graphs of compact urban street and canal patterns that we have studied is given in Table I. The spectral properties of finite Markov chains defined on these dual graphs are compared with those of a model example: a hypothetical village extended along one principal street and composed of N blind passes branching off it.

TABLE I. Some features of dual graphs of the studied compact city patterns: $|V|$ is the number of streets, $|E|$ the number of junctions (crossroads), and the graph diameter diam(G) is the maximal graph-theoretical distance between any two vertices of the dual graph. In the last column, the random target access time τ, the expected number of steps required to hit a node randomly chosen in the city from the stationary distribution π. The random target access time is independent of the starting point.

	$\|V\|$	$\|E\|$	diam(G)	τ
Rothenburg o.d.T.	50	115	13	545
Bielefeld downtown	50	142	14	551
Amsterdam canals	57	200	11	849
Venice canals	96	196	14	1550
Manhattan	355	3543	17	4557

A square mile of the New York city grid and a square mile pattern of the Venice street array have been discussed recently in [15,18–20]; however, to our knowledge, the canal patterns have never been subjected to a network analysis. The navigation efficiency in Manhattan streets has been studied in [16].

It is worth mentioning the importance of the implemented street identification principle for the conclusion on the degree statistics of dual city graphs. The comparative investigations of different street patterns performed in [6,15] implementing the ICN principle reveal scale-free degree distributions for the vertices of dual graphs. However, in [17] it has been reported that under the street-name approach the dual graphs exhibit small-world character, but scale-free degree statistics can hardly be recognized. The results on the probability degree statistics for the dual graphs of compact urban patterns analyzed in accordance with the street identification principle that we have described above are compatible with that of [17]. Compact city patterns do not provide

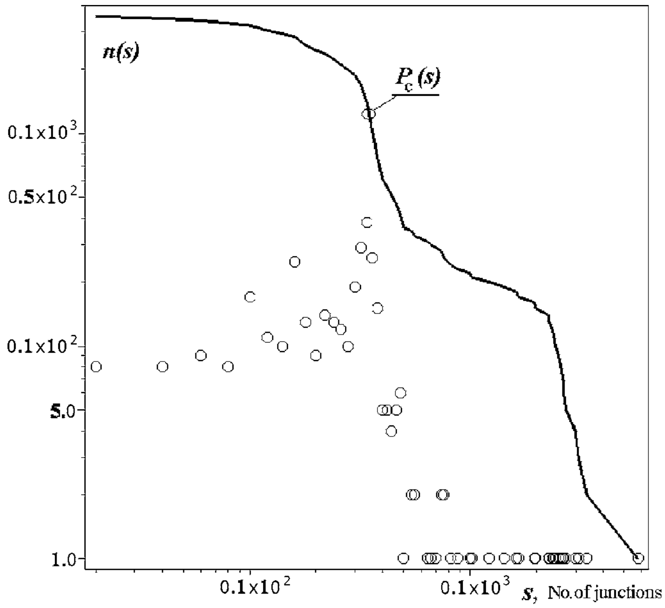

FIG. 1. The logarithm of the fractions of streets, $\ln n(s)$, via the logarithm of the number of their junctions, $\ln s$ in Manhattan (shown by points). The solid line represents the logarithm of the relevant cumulative distribution, $\ln P_c(s)$ [7].

us with sufficient data to conclude on the universality of degree statistics. It is remarkable that the probability degree distributions for the dual graphs correspondent to the compact city patterns are broad and have a clearly expressed maximum and a long right tail. The presence of a noticeable maximum in the probability degree distributions indicates that the structures of compact urban patterns are usually close to a regular one and that there is the most probable number of junctions an average street has in a given city. The long right tails of distributions correspond to the highly connected nodes of dual graphs, just a few "broadways," embankments, and belt roads crossing many more streets than an average street in the city. To give an example, in Fig. 1, we display the log-log plot of fractions of streets $n(s)$ via the number of their junctions s in Manhattan. These numbers are shown by points, and the solid line is for the relevant cumulative distribution $P_c(s)=\Sigma_{s'=s}^{\infty} n(s')$ [7].

II. RANDOM WALKS ON DUAL CITY GRAPHS: THE DESCRIPTION OF MODELS AND A SKETCH OF RESULTS

We consider a connected graph $G=(V,E)$ with $|V|=N$ nodes and $|E|=m$ undirected edges specified by its adjacency matrix A such that $A_{ij}=1$ if the node i is connected to j and $A_{ij}=0$ otherwise. A random walk starting at a node $v_0 \in V$ and traversing a sequence of random nodes $\{v_t\}$ as $t=0,1,\ldots$ is a Markov chain characterized by the matrix of transition probabilities $T=DA$, in which D is the diagonal matrix of inverse vertex degrees, $D_{ij}=k_i^{-1}\delta_{ij}$, where $k_i=\deg(i)$, the degree of vertex i in graph G (the number of junctions a street shares on the city plan). The transition matrix T meets the normalization condition $\Sigma_j T_{ij}=1$. We denote by $\pi^t \in \mathbb{R}^N$ the distribution of v_t in the Markov chain, $\pi_i^t=\Pr(v_t=i)$. Then the rule of the walk can be expressed by the simple equation

$$\pi^{t+1} = T^{\dagger}\pi^t, \tag{1}$$

where $T^{\dagger}$ is the transposed transition matrix. In the present paper, we discuss only undirected dual city graphs with symmetric adjacency matrices, $A_{ij}=A_{ji}$. Nowadays, certain driving directions are specified for the streets by traffic regulation polices, so that the relevant transportation lines appears to be directed. We do not consider them, limiting our present study only to the network of *pedestrian access*. The distribution of a current node in the random walk defined by Eq. (1) on undirected graphs after t steps tends to a well-defined *stationary* distribution $\pi_i=k_i/2m$ (which is uniform if the graph is regular) that is a left eigenvector of the transition matrix T, belonging to the largest eigenvalue 1 [12,21]. The spectrum of problem (1) is contained in the interval $[-1,1]$.

In Sec. III, we calculate and analyze the expected number of steps a random walker starting from node i makes before node j is visited (the access time) for all pairs of streets (canals) in compact urban patterns. The properties of access times can be used in order to estimate the accessibility of certain streets and districts by random walkers starting from the rest of city. In particular, the access times allows one to introduce the equivalence classes of nodes and to obtain a well-defined ordering of these equivalence classes, independent of any reference node. The nodes in the lowest class are difficult to reach but easy to get out of, while the nodes in the highest class are easy to reach but difficult to leave. We also discuss the random target access times and the distributions of mean access times in the compact cities. The latter characteristics can be used in order to detect ghettos (the groups of relatively isolated nodes) and estimate the accessibility of certain districts from the streets located in other parts of the city.

The problem of random walks (1) defined on finite graphs can be related to a diffusion process which describes the dynamics of a large number of random walkers. The associated Laplace operator is more convenient, since its spectrum is positive. Indeed, the eigenvalues in Eq. (1) can be negative, so that the spectral moments $\Sigma_i \lambda_i^{-n}$ could oscillate strongly for large networks. In contrast, the eigenvalues of the Laplacian are positive, which allows one to study its spectral properties by the powerful methods of statistical mechanics, in which characteristic functions $f(\lambda_i)$ defined on the spectrum $\{\lambda_i\}$ are considered.

The diffusion process is defined by the expectation number of random walkers, $\mathbf{n} \in \mathbb{R}^N$, and described by the equation

$$\dot{\mathbf{n}} = L\mathbf{n}, \tag{2}$$

in which the scaled Laplace operator L (symmetric) is defined by

$$L = \mathbf{1} - D^{-1/2}TD^{1/2}, \tag{3}$$

where $\mathbf{1}$ is the unit matrix. Let us note that L is related to the transition matrix T in Eq. (1) by $T=D^{1/2}(\mathbf{1}-L)D^{-1/2}$. All eigenvalues of L belong to the interval $\lambda_\alpha \in [0,2]$; the eigenvectors are normalized, $|\mathbf{n}^{(\alpha)}|=1$, and orthogonal, $\langle \mathbf{n}^{(\alpha)}\mathbf{n}^{(\beta)}\rangle = \delta_{\alpha\beta}$.

The analysis of spectral properties of the Laplace operator allows for detection of fine-scale dynamical modularity of the city which cannot be seen from the transition matrix T and provides us a tool for the estimation of the entire city stability with respect to occasional cuts of certain transportation lines breaking it up into dynamically isolated components. In Sec. IV, we consider the modes of the diffusion process defined on the dual city graphs and compute the coefficients of linear correlations between densities of random walkers that flow along the edges of dual city graphs in discrete steps. Specifying a certain correlation threshold, we detect the groups of nodes traversed by the essentially correlated flows of random walkers. The structures of essentially correlated flows and their appearance are the individual characteristics of a city. We measure the quantity and extension of clusters of nodes sharing the essentially correlated flows of random walkers by an information parameter reflecting the complexity of random walk traffic.

In Sec. V, we discuss the statistical mechanics of so-called *lazy random walks* specified by the parameter $0<\beta\leq 1$. In the model of lazy random walks, an agent located at a node v moves to a neighboring node with probability βk_v^{-1}, but rests modeless with probability $(1-\beta)$. We compute the well-known thermodynamic quantities (the internal energy, entropy, the free energy, and pressure) describing the macroscopic states of an ensemble of lazy random walkers on the dual graphs of compact urban patterns. We provide a detailed interpretation for each thermodynamic parameter in the context of random walks.

III. ACCESS TIMES IN COMPACT CITY STRUCTURES

The important characteristic of random walks defined on finite graphs is the *access time* H_{ij}, the expected number of steps before node j is visited, starting from node i [12]. The elements of the matrix H_{ij} are computed for each pair of nodes i,j following the formula

$$H_{ij} = 2m\sum_{s=2}^{N} \frac{1}{1-\mu_s}\left(\frac{\varphi_{sj}^2}{k_j} - \frac{\varphi_{si}\varphi_{sj}}{\sqrt{k_i k_j}}\right), \qquad (4)$$

in which $\mu_1 = 1 > \mu_2 \geq \cdots \mu_N \geq -1$ are the eigenvalues and φ_s are the relevant eigenvectors of the symmetric transition matrix $D^{-1/2}TD^{1/2}$.

The access time from i to j may be different from the access time from j to i, $H_{ij}\neq H_{ji}$, even in a regular graph. A deeper symmetry property of access times for undirected graphs was discovered in [22],

$$H_{ij} + H_{jk} + H_{ki} = H_{ik} + H_{kj} + H_{ji}, \qquad (5)$$

for every three nodes in G. This property allows for the ordering of nodes in the graph with respect to their accessibility for random walkers. It has been pointed out in [12] that the nodes of any graph can be ordered, so that if i precedes j, then $H_{ij}\leq H_{ji}$. This ordering is not unique, but one can partition the nodes by putting i and j in the same equivalence class if $H_{ij}=H_{ji}$ and obtain a well-defined ordering of the equivalence classes, independent of any reference node. The nodes in the lowest class are difficult to reach but easy to get out of, while the nodes in the highest class are easy to reach but difficult to leave. If a graph has a vertex-transitive automorphism group, then $H_{ij}=H_{ji}$ for all $i,j\in G$. The random target identity [21]

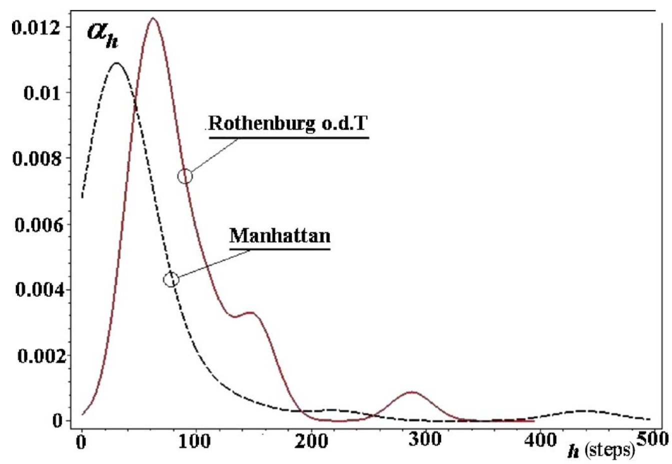

FIG. 2. (Color online) The distributions of mean access times h to the streets in Manhattan (dashed line) and Rothenburg ob der Tauber (solid line). The distribution α_h for Rothenburg o.d.T. exhibits a local maximum at relatively long access times (300 steps), indicating the presence of a number of low-accessible streets in the town.

$$\sum_j \pi_j H_{ij} = \text{const} \qquad (6)$$

states that the expected number of steps (the random target access time τ) required to reach a node randomly chosen from the stationary distribution π is a constant, independent of the starting point of the given graph G. The values of random target access times grow with the size of graphs and are very sensitive to their structures. Their values for compact urban structures are given in the last column of Table I.

The properties of access times can be used in order to estimate the accessibility of certain streets and districts by random walkers starting from the rest of city. Computations of access times to the streets in the studied compact urban structures convinced us that for any given node i the access times to it, H_{ij}, change with j inferentially in comparison with their typical values, and therefore, the mean access time

$$h_i = \frac{1}{N}\sum_{j=1}^{n} H_{ij} \qquad (7)$$

can be considered as a good parameter for estimating the accessibility of a street by random walkers. Distributions of mean access times in the city, α_h, can be considered as the estimations of its connectedness. In particular, it helps to detect the ghettos, the groups of streets almost isolated (in the dynamical sense) from the rest of town. In Fig. 2, we present together the distributions of mean access times to the streets in Manhattan (dashed line) and Rothenburg o.d.T. (solid line). The distribution for Rothenburg o.d.T. exhibits a

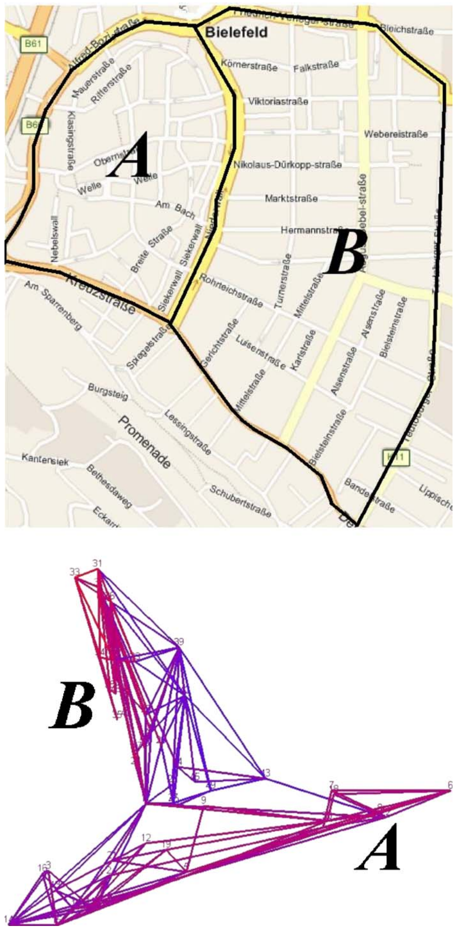

FIG. 3. (Color online) The city map of Bielefeld downtown (a) is presented together with its 3D representations of the dual graph (b). The A part keeps its original structure (founded in the 13th–14th centuries); part B which had been redeveloped in the 14th century. The (x_i, y_i, z_i) coordinates of the ith vertex of the dual graph in three-dimensional space are given by the relevant ith components of three eigenvectors $u^{(2)}$, $u^{(3)}$ and $u^{(4)}$ of the adjacency matrix A of the dual graph.

local maximum at relatively long access times (≈300 steps), indicating that there are a number of low accessible streets in the town, a ghetto.

The distribution of mean access times in the downtown of Bielefeld is of essential interest since it is comprised of two structurally different parts [see Fig. 3(a)]. Part A keeps its original structure (founded in the 13th–14th centuries), while part B had been subjected to the partial redevelopment in the 19th century [Fig. 3(a)].

It is important to mention that the city districts constructed in accordance with different development principles and in different historical epochs can be easily visualized on the dual graph of the city. In Fig. 3(b), we show a three-dimensional (3D) representation of the dual graph of the Bielefeld downtown. The (x_i, y_i, z_i) coordinates of the ith vertex of the dual graph G in 3D space are given by the relevant ith components of three eigenvectors $u^{(2)}$, $u^{(3)}$, and $u^{(4)}$ of the adjacency matrix A_G (which does not coincide with the transition matrix T). These eigenvectors correspond to the second, third, and fourth largest (in absolute value) eigenvalues of A_G [23]. The 3D dual graph of Bielefeld displays clearly a structural difference between A and B parts: in 3D representation, the relevant subgraphs are located in the orthogonal planes [Fig. 3(b)]. Sometimes other symmetries of dual graphs can be discovered visually by using other triples of eigenvectors if the number of nodes in the graph is not too large.

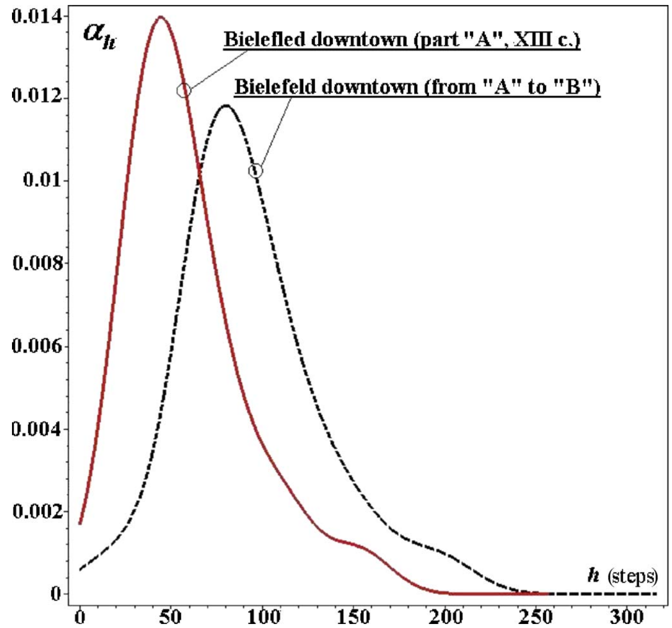

FIG. 4. (Color online) The distributions of mean access times h to the streets located in the medieval part A starting from those located in the same part of Bielefeld downtown, from A to A (solid line). The dashed line presents the distribution of mean access times to the street located in the modernized part B starting from the medieval part A (from A to B). An average, in takes a longer time to reach the streets located in B starting from A.

In Fig. 4, we display the distributions of mean access times h to the streets located in the medieval part A starting from those located in the same part of Bielefeld downtown, from A to A (solid line). It has been computed by averaging H_{ij} over $i,j \in A$ in Eq. (7). The dashed line presents the distribution of mean access times to the streets located in the modernized part B starting from the medieval part A (from A to B, $i \in A$ and $j \in B$). One can see that in average in takes longer time to reach the streets located in B starting from A. A similar behavior is demonstrated by the random walkers starting from B (see Fig. 5): on average, it requires longer time to leave a district for another one. Study of random walks defined on the dual graphs helps to detect the quasi-isolated districts of the city.

IV. DYNAMICAL MODULARITY IN COMPACT CITY STRUCTURES

Dynamical modularity is a particular division of the graph into groups of nodes on which the certain modes of diffusion process are localized. It can be detected by analyzing spectral properties of the relevant dual graph that is of the spectrum of some differential operator defined on it [24]. In particular,

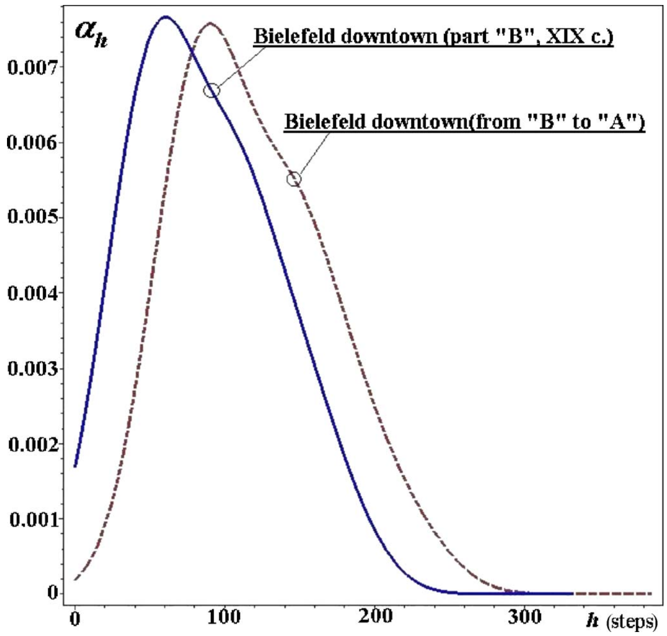

FIG. 5. (Color online) The distributions of mean access times h to the streets located in the B part starting from B (solid line). The dashed line presents the distribution of mean access times to the street located in the A part starting from B.

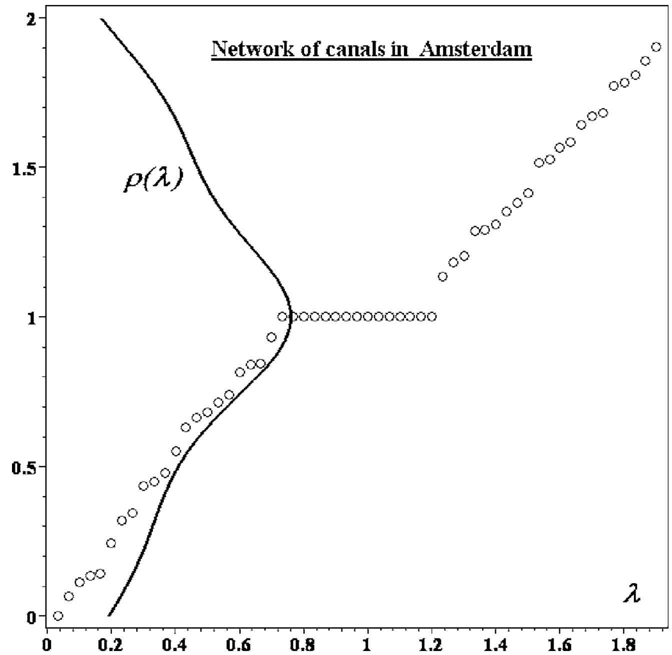

FIG. 7. The eigenvalues of the Laplace operator (3) defined on the dual graph of Amsterdam canal network together with the continuous approximation of its density.

for the undirected graphs, it is convenient to consider differential elliptic self-adjoint operators with a positive spectrum. We have studied the Laplace operator defined by Eq. (3).

The eigenvalues of Laplace operator (3) along with the continuous approximations of their densities for the dual graphs of compact urban patterns are shown in Figs. 6–8. The similarity of the spectra allows us to divide the studied compact urban patterns into three categories: (i) medieval cities, Bielefeld (Fig. 6) and Rothenburg o.d.T.; (ii) canal patterns, Venice (Fig. 7) and Amsterdam; (iii) spectra with a highly degenerated eigenmode ($\lambda = 1$), Manhattan (Fig. 8).

It is important to note that the densities of eigenvalues for *compact* urban patterns differ dramatically from those computed for classical random graphs of the Erdös-Rényi model [25,26], deviate from the semicircular law, and the densities found for the scale-free random-tree-like graphs in [27]. In [28], the density of eigenvalues for the Internet graph on the autonomous system (AS) level is computed. The Internet spectrum on the AS level is broadly distributed with two symmetric maxima and similar to the eigenvalue density of random scale-free networks. In contrast, the spectral density

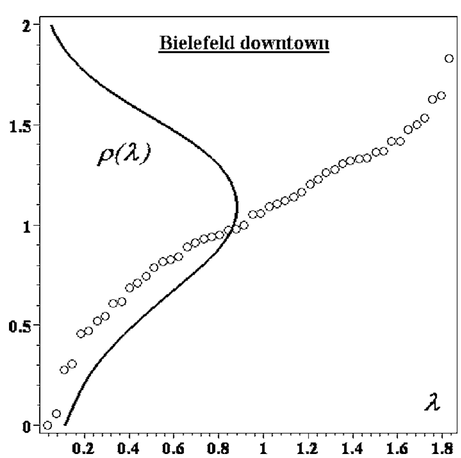

FIG. 6. The eigenvalues of the Laplace operator (3) defined on the dual graph of Bielefeld together with the continuous approximation of its density.

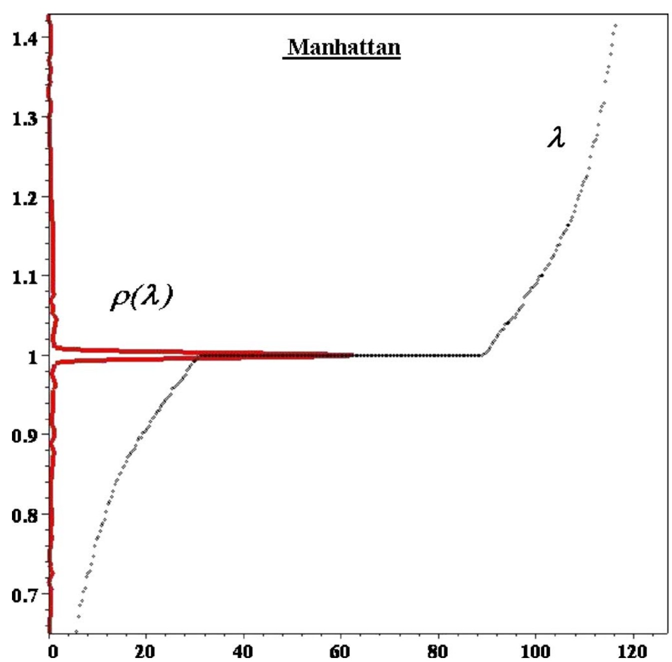

FIG. 8. (Color online) The eigenvalues of the Laplace operator (3) defined on the dual graph of Manhattan together with the continuous approximation of its density.

for the compact city samples are bell shaped and tend to turn into a sharp peak localized at $\lambda=1$ due to the highly degenerate one-mode which would score a valuable fraction of eigenvalues (48% for Manhattan). One-modes come in part from pairs of streets of minimal connectivity branching of either "broadways" or belt roads. These structures are over-represented in compact city patterns.

The slowest modes of diffusion process (2) allow one to detect the city modules characterized by the individual accessibility properties. The primary feature of the diffusion process in compact urban patterns is the flow between the dominant pair of city modules: the broadways and the relatively isolated streets remote from the primary roads.

Due to the proper normalization, the components of the eigenvectors $\mathbf{n}^{(\alpha)}$ play the role of the participation ratios (PR's) which quantify the effective numbers of nodes participating in a given eigenvector with a significant weight. This characteristic has been used in [28] and by other authors to describe the modularity of complex networks. However, in a majority of highly degenerate modes, PR is not a well-defined quantity (since the different vectors in the eigenspace corresponding to the degenerate mode would obviously have different PR's).

As time advances the distribution of random walkers approaches a steady state $n_i^{\infty} \propto k_i$, in which all diffusion currents are balanced. It corresponds to the principal eigenvector $n_i^{(1)}$ related to the smallest eigenvalue of L. The relaxation processes toward the steady state are described by the remaining eigenvectors $\mathbf{n}^{(\alpha)}$, $\alpha>1$, with the characteristic decay times $\tau^{(\alpha)}$, such that $\exp(-1/\tau^{(\alpha)})=\lambda_\alpha$. The second smallest eigenvalue of the scaled Laplace operator (3) is related to the graph diameter, $\mathrm{diam}(G) \leq -\ln(N-1)/\ln(\lambda_2)$, the maximum distance between any two vertices in the graph. It is also related to the Fiedler vector [29] describing the algebraic connectivity of the graph. Namely, let us consider the components of the eigenvector $\mathbf{n}^{(2)}$ corresponding to the second smallest eigenvalue λ_2 for the scaled Laplacian (3) defined on a connected graph $G(V,E)$. Define $V_1=\{v \in V: n_v^{(2)}<0\}$ and $V_2=\{v \in V: n_v^{(2)} \geq 0\}$; then, the subgraphs induced by V_1 and V_2 are connected.

In general, each nodal domain on which the components of the eigenvector $\mathbf{n}^{(\alpha)}$ does not change sign refers to a coherent flow (characterized by its decay time $\tau^{(\alpha)}$) of random walkers toward the domain of alternative sign. The nodal domains participate in the different eigenmodes as one degree of freedom, and therefore their total number is important for detecting the dynamical modularity of city networks. It is known from [30] that the eigenvector $\mathbf{n}^{(\alpha)}$ can have at most $\alpha+m_\alpha-1$ strong nodal domains (the maximal connected induced subgraphs, on which the components of eigenvectors have a definite sign) where m_α is the multiplicity of the eigenvalue λ_α, but not fewer than two strong nodal domains ($\alpha>1$) [31]. However, the actual number of nodal domains can be much smaller than the bound obtained in [31]. In the case of degenerate eigenvalues, the situation becomes even more difficult because this number may vary considerably depending upon which vector from the m_α-dimensional eigenspace of the degenerate eigenvalue λ_α is chosen. A fragment of the nodal matrix for the Chelsea village [Fig. 9(a)] in Manhattan island is shown on Fig. 9(b): the components of all eigenvectors localized on the nodal domains displayed in white (black) have always positive (negative) sign.

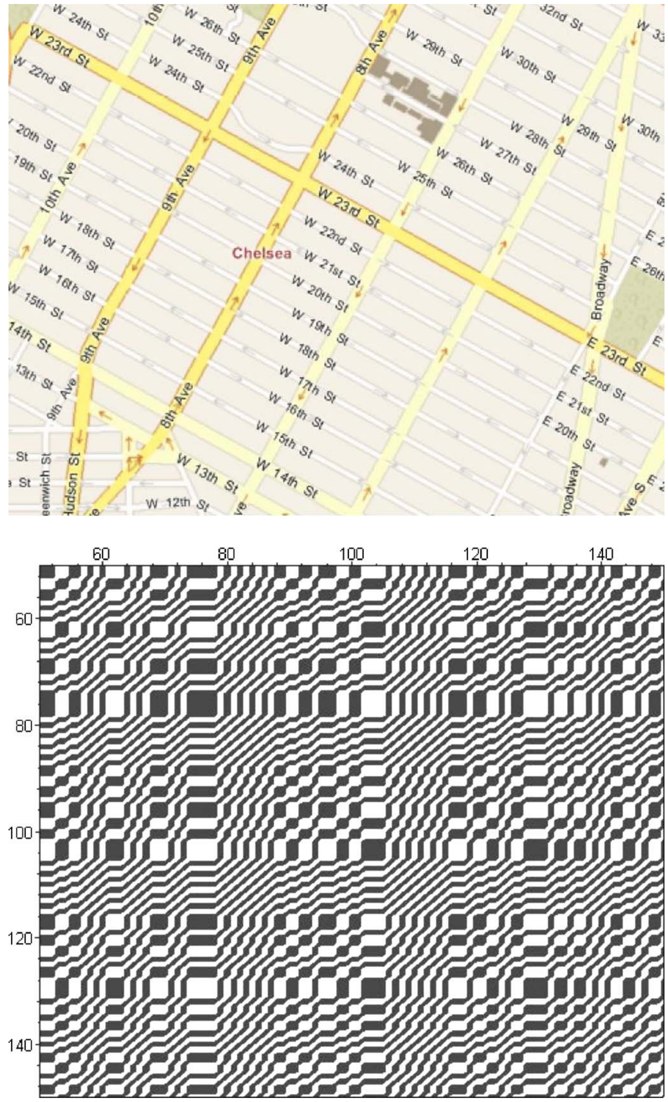

FIG. 9. (Color online) (a) Plan of Chelsea village in Manhattan and the matrix plot of its nodal domains (b). All eigenvectors localized on the nodal domains displayed in white (black) have always positive (negative) sign.

We investigate the statistics of components of eigenvectors $\mathbf{n}^{(\alpha)}$ localized on a given street, $\{n_i^{(\alpha)}\}_\alpha$. To uncover a fine modular structure of compact city patterns, we have computed the linear correlation coefficients between the lists of eigenvector components for all pairs of streets in a city,

$$C_{ij} = \frac{\mathrm{Cov}(\{n_i^{(\alpha)}\}_\alpha, \{n_j^{(\alpha)}\}_\alpha)}{\sqrt{\mathrm{Var}(\{n_i^{(\alpha)}\}_\alpha)\mathrm{Var}(\{n_i^{(\alpha)}\}_\alpha)}}. \tag{8}$$

The linear correlation measures how well a linear function explains the relationship between two data sets. The correlation is positive if an increase in the eigenvector components localized on a given street corresponds to an increase in those related to other streets and negative when an increase in one corresponds to a decrease in the other.

The study of correlations between the components of eigenvectors allows for a precise recovering of all dynamical modules in a city, street by street. By tuning the sensitivity threshold $\kappa > 0$, one can detect the groups of streets characterized by significant pairwise correlations $C_{ij} \geq \kappa$. In case of a degenerate eigenmode, the whole bunch of streets of minimal accessibility joins a correlation cluster at once. Investigating the size of clusters characterized by the significant pairwise correlations between the eigenvectors components, we have found that it is very sensitive to the threshold value κ. Pairs of correlated streets can be detected for $\kappa \leq \kappa_c$, below the critical value κ_c individual for each city. For instance, the strongest correlations in Manhattan island ($\kappa \geq 0.16$) are observed between Bowery Street (Chinatown) and 6th Avenue (Greenwich Village) and between Lafayette Street and Mott Street (Chinatown). By reducing κ just by 1%, we immediately get many new correlated pairs and (South FDR Drive, Allen Street), (Greene Street (Soho), Avenue D), (Crosby Street (Little Italy), Avenue D), (Centre Street, Elizabeth Street), (Allen Street, 6th Avenue), (Lafayette Street, Avenue B), (Broadway, West Street) among others. In most cases the streets in correlated pairs have the same driving directions although the driving direction data have not been initially used in the adjacency matrices. Reducing the value of sensitivity threshold, one can detect the triples of correlated streets and more structurally complicated dynamical modules. Then the correlated clusters merge, exhibiting a sharp phase transition into a giant connected component for essentially small κ.

The appearance of correlated k-tuples and their total numbers at given κ are by no means random and encode important *information* on the city connectedness. The complexity of dynamical modularity can be measured by a quantity of information encoded by the number of various k-tuples of essentially correlated nodes at different values of the correlation threshold κ in the following way. Since the probability that a connected correlated k-tuple appears in a *random* labeled graph of N nodes is

$$p_k(N) = \binom{N}{k}^{-1},$$

its appearance corresponds to some quantity of information $I = -p_k(N)\log_2 p_k(N)$. For a mesh of values $\kappa > 0$, we counted the total numbers of all correlated k-tuples appearing in each city, $\mathcal{N}_k(\kappa)$, and then found the total amount of information $\mathcal{I}(\kappa)$ encoded by the dynamical modularity in each studied city pattern,

$$\mathcal{I}(\kappa) = -\sum_{k\text{-tuples}} \mathcal{N}_k p_k(|V|)\log_2 p_k(|V|). \tag{9}$$

It is worth mentioning that in the absence of correlations as well as in the case when *all* fluxes of random walkers though nodes of the dual city graph are correlated, no information can be encoded, and therefore $\mathcal{I} = 0$. The results on the comparative information analysis of dynamical modularity of the compact urban structures are displayed in Figs. 10–12.

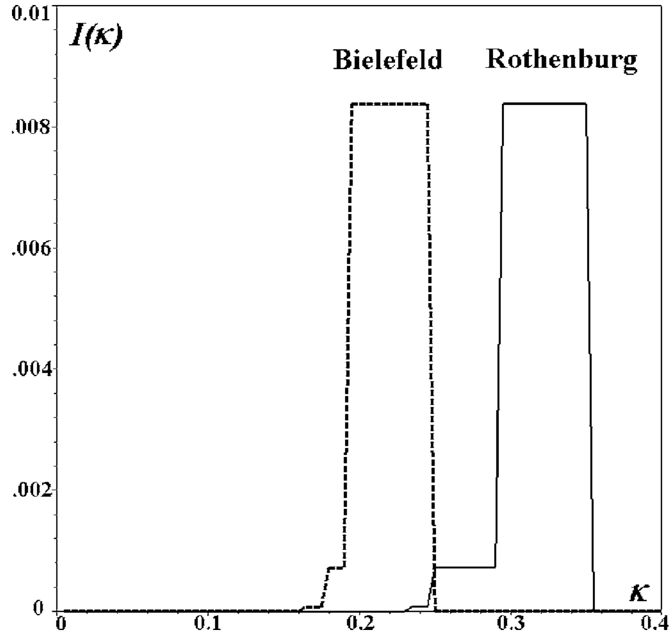

FIG. 10. Quantity of information (bits) encoded by the dynamical modularity of Rothenburg and Bielefeld via the threshold of essential correlations, $0 \leq \kappa \leq 1$.

Information (via the threshold of correlations, κ) calculated for Rothenburg and Bielefeld (subjected to a partial redevelopment) are almost equal (Fig. 10). Both medieval cities contain a number of streets characterized by relatively highly correlated traffic (they are the belt roads encircling the cities along their fortress walls), while traffic along the streets close to the city centers appears to be less correlated, although *all* streets of low accessibility join the correlated

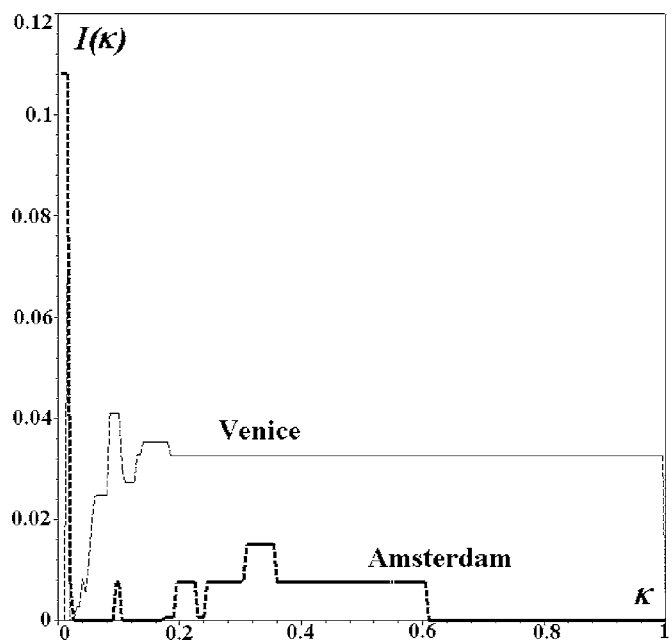

FIG. 11. Quantity of information (bits) encoded by the dynamical modularity of the Venice and Amsterdam canal networks via the threshold of essential correlations, $0 \leq \kappa \leq 1$.

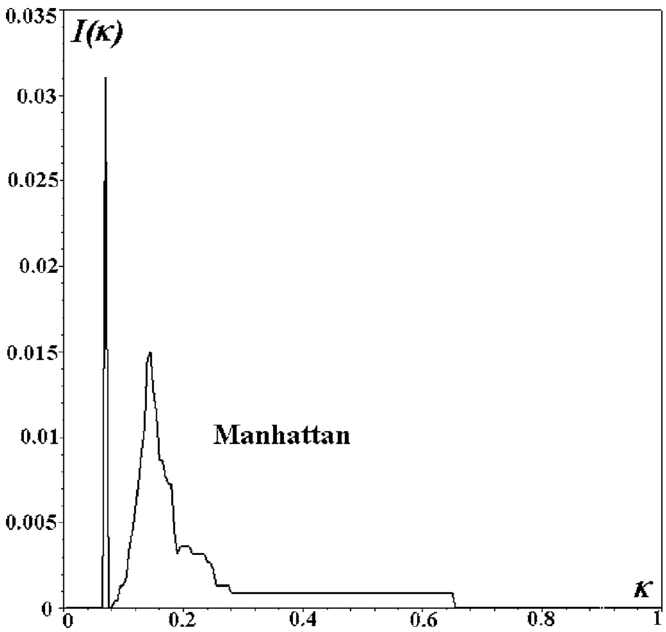

FIG. 12. Quantity of information (bits) encoded by the dynamical modularity of Manhattan via the threshold of essential correlations, $0 \leq \kappa \leq 1$.

cluster at once at some level κ; then, the information on the dynamical modularity turns to zero.

Information profiles obtained for the Venice and Amsterdam canal networks look similar (see Fig. 11). However, it seems that information in a message on possible correlations of gondoliers traffic in Venice could be much more valuable than a cruise schedule along the Amstel embankments. The triggering between different information states displayed in Fig. 11 corresponds to the merging of diverse correlated clusters into larger correlated modules. The information peaks located close to $\kappa=0$ indicate a phase transition to a giant correlated component which covers most of the nodes in the networks, but *not all*.

The scale of the graph shown in Fig. 12 is incompatible with those of Figs. 10 and 11 since the dynamical modularity in Manhattan island (one of the most regular city grids in the world) contains essentially less information (it comes primarily from the correlated clusters risen in the region of Central Park) than that in the ancient cities.

V. STATISTICAL MECHANICS OF LAZY RANDOM WALKS IN COMPACT CITY STRUCTURES

A number of different "measures" quantifying the various properties of complex networks have been proposed in so far in a wide range of studies in order to distinguish the groups of nodes and shed light on the relations between them. Some measures can be computed directly from the graph adjacency matrix: *likelihood* [32], *assortativity* [33], *clustering* [34], *degree centrality* [35–37], *betweeness centrality* [37], *link value* [38], *structural similarity* [39], and *distance* (counting the number of paths between vertices) [40]. Other measures (concerned with the networks embedded into Euclidean space) involve the lengths of links or the true Euclidean distances between nodes: *closeness centrality* [35,41], *straightness centrality* [42], [43], *expansion* [38], and *information centrality* and *graph efficiency* [43,44]. A good summary of the several centrality measures can be found in [20,45] for the Internet-related measures. The list of available measures is still far from being complete, new measures appearing together with any forthcoming network model. It is also worth adding some spectral measures (concerned the eigenvalues of graph adjacency matrix): *subgraph centralization* [46], *subgraph centrality* [47], *network bipartivity* [48], and many others.

In the present section, we study the statistics of flows of lazy random walkers roaming in the compact city patterns. The random walkers have no mass and do not interact with each other, so that they do not contribute to the energy or to the momentum transfer. Nevertheless, their flows have the nontrivial thermodynamic properties induced by the complex topology of streets and canals they flow along. The obvious advantage of statistical mechanics is that the mathematical objects we introduce and all relations between various statistical quantities are well known in the framework of thermodynamic formalism.

In the following, we use the inverse temperature parameter $\beta > 0$ which can be considered either as an effective time scale in the problem (the number of streets a walker passes in one time step) or as the laziness parameter defined in Sec. II. Since the eigenvalues λ_α of the scaled Laplace operator (2) are positive and bounded, one can define for them three well-known spectral functions: (i) the heat kernel (the partition function),

$$K(\beta) = \mathrm{Tr}\,\exp(-\beta L) = \sum_{\alpha=1}^{N} e^{-\beta\lambda_\alpha}, \tag{10}$$

converging as $\beta \geq 0$, for $N \to \infty$, (ii) the spectral zeta function (the spectral moments)

$$\zeta(s) = \sum_{\alpha=1}^{N} \lambda_\alpha^{-s}, \tag{11}$$

and (iii) the spectral density

$$\rho(\lambda) = \sum_{\alpha=1}^{N} \delta(\lambda - \lambda_\alpha). \tag{12}$$

These spectral functions are related to each other by the Laplace transformation

$$K(\beta) = \int_0^\infty d\lambda e^{-\beta\lambda}\rho(\lambda) \tag{13}$$

and by the Mellin transformation (up to the Γ function) [50]

$$\zeta(s) = \frac{1}{\Gamma(s)}\int_0^\infty \beta^{s-1}K(\beta)d\beta. \tag{14}$$

Furthermore,

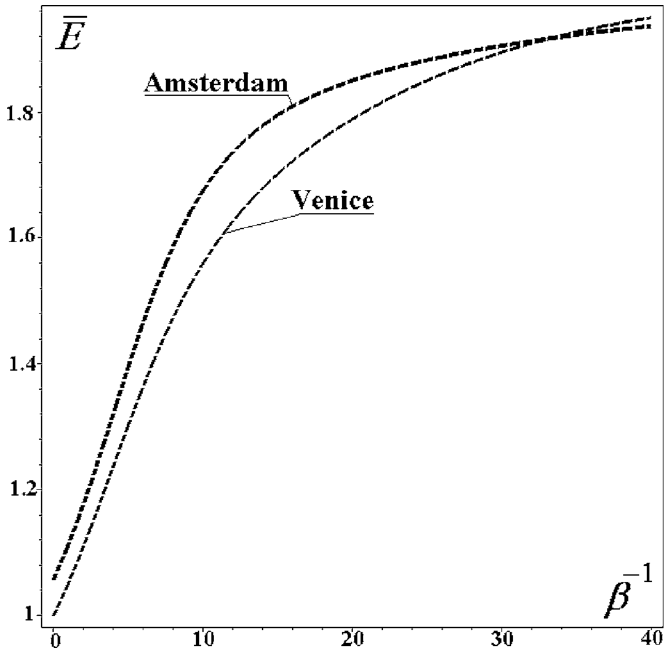

FIG. 13. The growth of "internal energy" in the models of lazy random walks with "temperature" β^{-1} (the inverse parameter of lazy random walks) for the canal networks of Amsterdam and Venice.

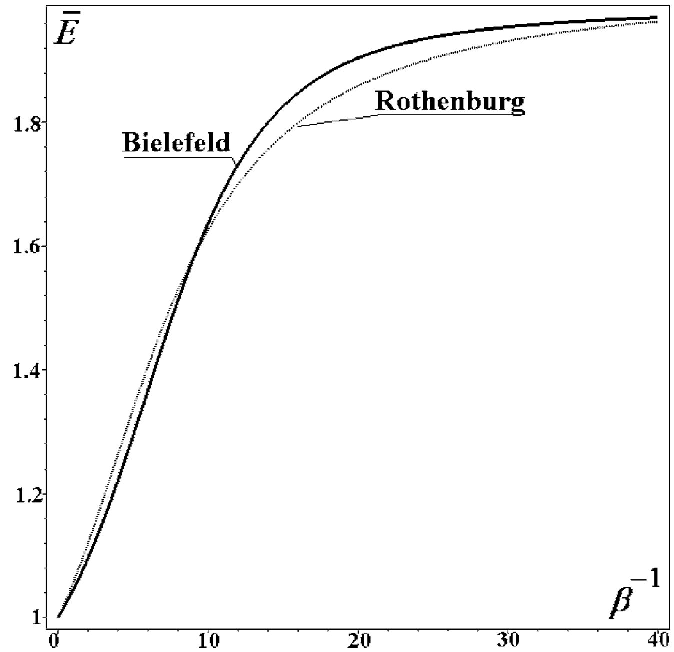

FIG. 14. The growth of "internal energy" in the models of lazy random walks with "temperature" β^{-1} (the inverse parameter of lazy random walks) for the medieval cities Rothenburg o.d.T. and Bielefeld.

$$\zeta(s) = \int_{-\infty}^{\infty} \rho(\lambda)\lambda^{-s} d\lambda. \qquad (15)$$

Let us note that since $0<\lambda_\alpha<2$, we immediately obtain bounds for the spectral moments,

$$\zeta(-n) < N \times 2^n, \quad \zeta(n) > N \times 2^{-n}.$$

The partition function $K(\beta)$ meets the conditions of Bernstein's theorem [49]:

$$(-1)^n K^{(n)}(\beta) \geq 0, \quad \beta > 0, \quad n \geq 1,$$

and therefore defines a unique non-negative measure $d\mu(\beta) = K(\beta)d\beta$ on $\mathbb{R}_+$ providing solutions of the Hamburger and Stiltjes moment problems [50]. Concerning the list of "measures" given above, it is worth noting that, in principle, any of spectral functions (10)–(12) at a given temperature β^{-1} can be used as a measure of some quantities relevant to a certain lazy random walk model.

The ensemble of random walkers on the graph $G(V,E)$ can be characterized by the following *macroscopic* quantities: (i) the internal (averaged) energy, $\bar{E}=-\partial_\beta \ln K(\beta)$, (ii) the entropy $S=\ln K(\beta)+\beta\bar{E}$, (iii) the free energy, $F=\beta^{-1}\ln K(\beta)$, and (iv) the pressure $P=\beta^{-1}\partial_{|V|}\ln K(\beta)$. In the last equality, we have considered the differential with respect to the graph size as $d|V|=\rho(\lambda)d\lambda$. The thermodynamics of compact urban city patterns is presented in Figs. 13–21 and in Table II. The collected data give an insight into the various aspects of *averaged* street (canals) accessibility in a given city.

Due to the complicated topology of streets and canals, the flows of random walkers exhibit spectral properties similar to those of a thermodynamic system characterized by a nontrivial internal energy (Figs. 13–15). It grows with temperature (the inverse parameter of lazy random walks) β^{-1} (albeit still bounded in the interval $[0,2]$). As usual, the absolute value of the internal energy relevant to a given city cannot be

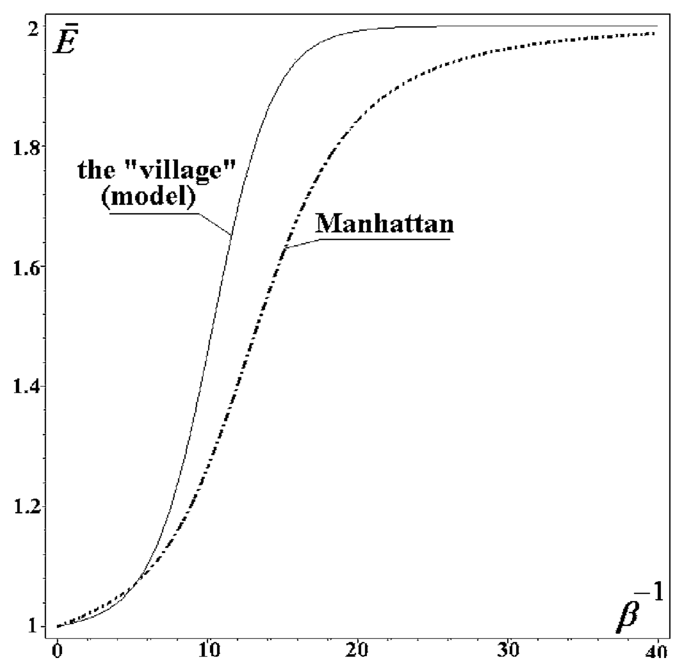

FIG. 15. The growth of "internal energy" in the models of lazy random walks with "temperature" β^{-1} (the inverse parameter of lazy random walks) for Manhattan and the theoretical example of a "village" extended along the only principal street.

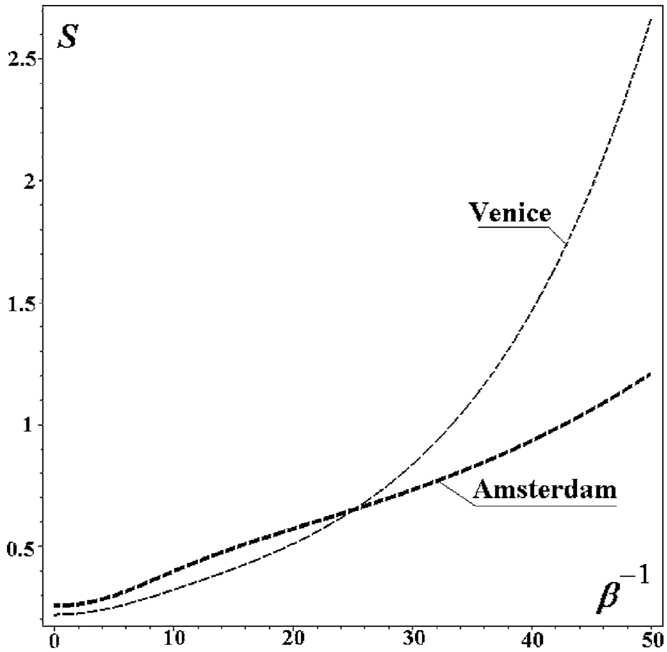

FIG. 16. The entropy curves via the inverse parameter of lazy random walks, β^{-1}, for the canal networks of Venice and Amsterdam.

precisely measured, but we can measure its difference in any temperature interval. In principle, the slopes of internal energy curves are steeper (the internal energy grows faster with β^{-1}) in modern cities with a quite regular grid like structure, in which any street has a relatively good accessibility.

In thermodynamics, entropy is an extensive state function that accounts for the effects of irreversibility in thermodynamic systems. It describes the number of possible microscopic configurations of the system. In the problem of finite

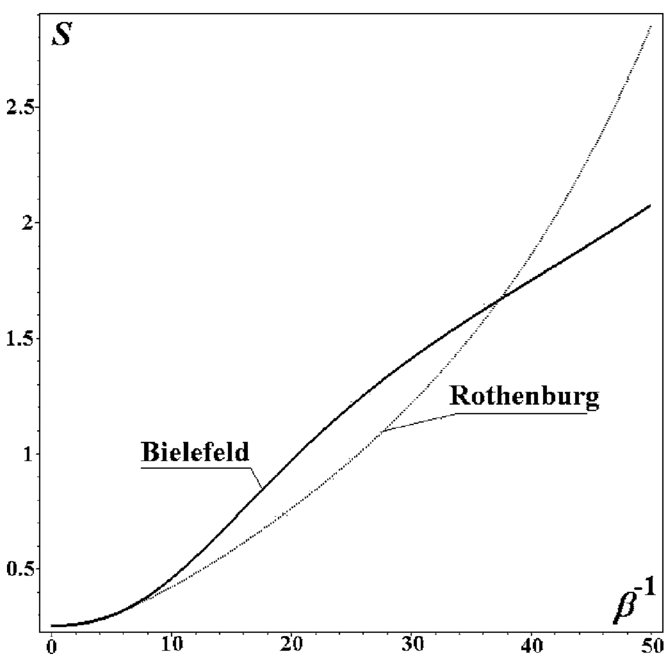

FIG. 17. The entropy curves via the inverse parameter of lazy random walks, β^{-1}, for Bielefeld and Rothenburg o.d.T.

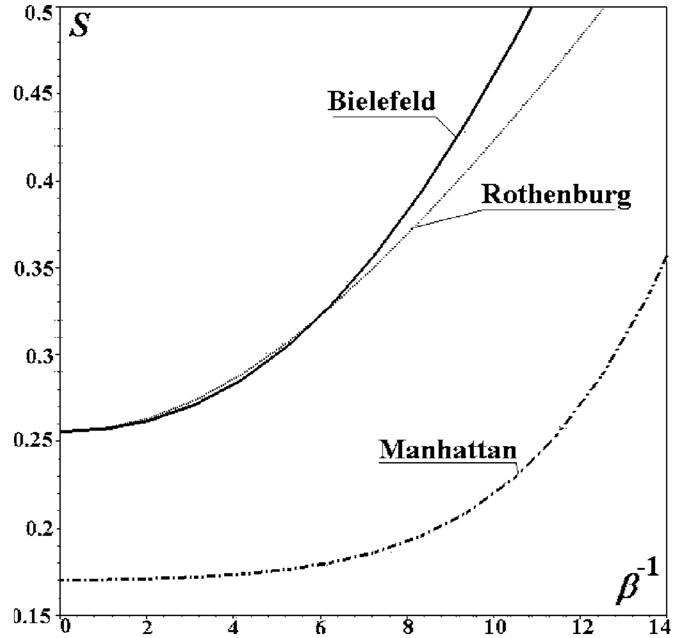

FIG. 18. The entropy curves via the inverse parameter of lazy random walks, β^{-1}, for Bielefeld, Rothenburg, and Manhattan.

random walks, its value quantifies the diversity of flows which can be detected in the city at a given temperature and grows with temperature. In Figs. 16–18, we display the entropy curves via β^{-1}. It is important to mention that the improvement of street accessibility causes a decrease of entropy growth rates for large temperatures (small β). The entropy of random walker flows as well as its growth rate in Manhattan is less than in any other compact city structure we studied.

Due to the numerous junctions and the highly entangled meshes of city streets and canals, the random walkers lose

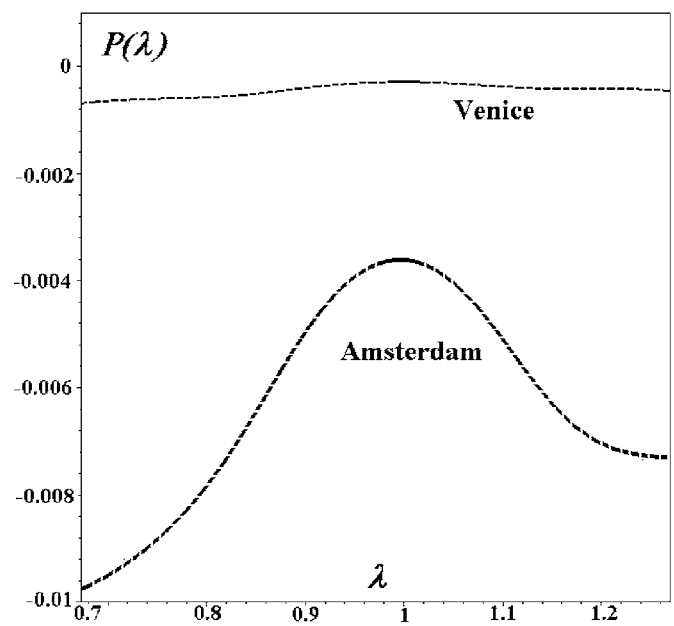

FIG. 19. The comparison of pressure spectra $P(\lambda)$ acting on the flows of random walkers with eigenmodes λ into Amsterdam and Venice.

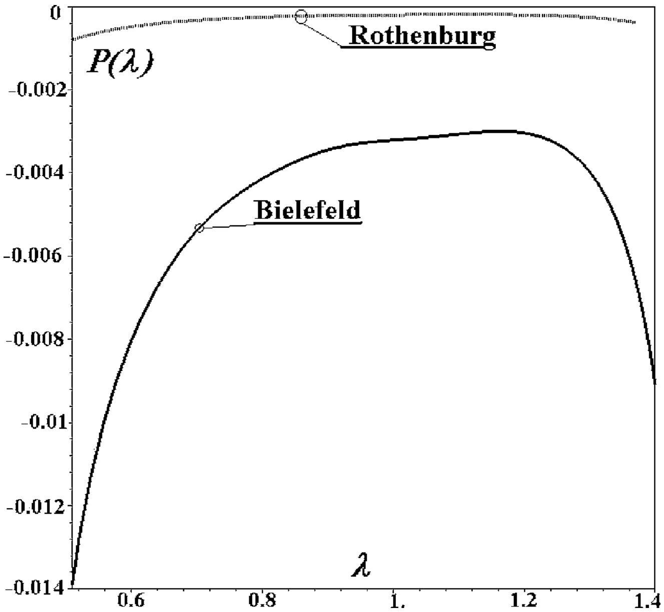

FIG. 20. The comparison of pressure spectra $P(\lambda)$ acting on the flows of random walkers with eigenmodes λ into Bielefeld and Rothenburg.

their way and it takes a long time for them to cross a city roaming randomly along the streets. This can be interpreted as an effect of a slight negative pressure involving the random walkers into the city. It is obvious that the strength of pressure should vary from one district to another within a city and can be different for the different modes of the diffusion process which have no rigorous bind to the city administrative units being localized on certain groups of streets and canals. We have computed the pressure spectra $P(\lambda)$, forcing the flows of random walkers with eigenmodes λ into

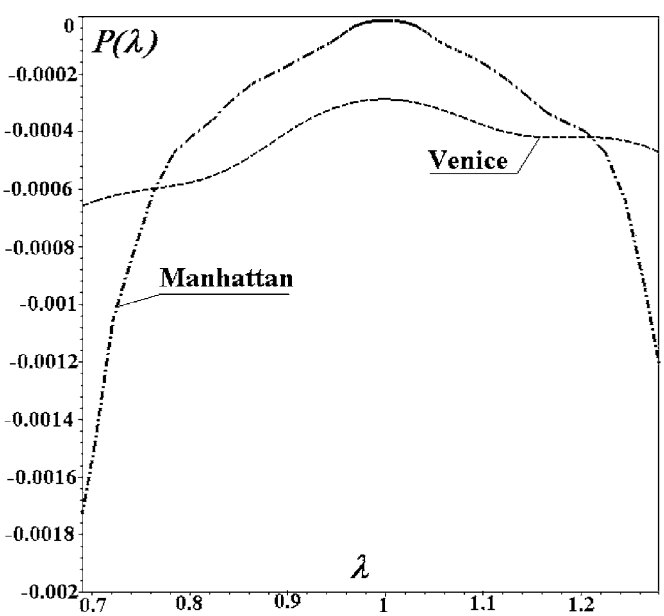

FIG. 21. The comparison of pressure spectra $P(\lambda)$ acting on the flows of random walkers with eigenmodes λ into Manhattan and Venice.

the compact city structures (see Figs. 19–21). Generally speaking, the more junctions a city has, the stronger is the drag force: despite Venice having more canals than Amsterdam, the number of junctions between them is less than in Amsterdam, and therefore the negative pressure in the latter city is stronger (Fig. 19). The numbers of streets in Rothenburg and in the downtown of Bielefeld are equal, but there are more crossroads in Bielefeld than in Rothenburg (Fig. 20).

One can see that pressure profiles have maxima close to $\lambda=1$ (corresponding to a minimal drug force) for Amsterdam, Venice, and Manhattan. It calls for the idea of a "transparency corridor"—i.e., a sequence of streets and junctions (on which the relevant eigenmodes are localized) along which the city can be crossed in minimal time. In Manhattan, the pressure profile is almost zero at $\lambda=1$. The eigenmode $\lambda=1$ is degenerated due to multiple junctions of low-accessible streets indistinguishable with respect to the random walker dynamics to Broadway and the FDR Drive. In Venice, the minimal pressure is achieved on the Grand Canal and Giudecca Canal, and it is due to Het IJ and Amstel river in Amsterdam.

In Table II, we have sketched the figures for the internal energy, entropy, the free energy, and heat capacity for all studied cities at $\beta=1$ (for the usual random walks model), for the purpose of comparison. We also gave the location of maxima for the heat capacity profiles.

VI. DISCUSSION AND CONCLUSION

We have studied the finite Markov chain processes defined on the dual graphs of compact urban patterns. The traffic of random walkers, indeed, does not describe the actual traffic conditions in the city patterns that we have analyzed, but concerns their topological properties giving a sense to the notions of accessibility and modularity. Our approach can be readily used in order to investigate the connectedness and efficiency of transportation lines in different complex networks.

The methods developed in graph theory and in probability theory give us a detailed picture of local and global properties of city structures. Dynamical modularity representing the localization of dynamical modes on certain streets and canals cannot be detected either from the adjacency matrix of a graph or from the transition probability matrix. We have studied the random walks of a large number of particles having no masses introduced in the dual graphs of compact cities and investigated their properties by means of statistical mechanics. The spectrum of the Laplace operator is broadly distributed for compact city patterns and has a bell shape, in general. However, it turns into a sharp peak if there are a number of low-accessible street branching of a prime street in a city.

To detect the dynamical modularity precisely, we have computed the linear correlation coefficients between the lists of eigenvector components for all pairs of streets (canals) in the cities. Tuning the sensitivity threshold of correlations, one can detect the various modules of essentially correlated streets. The complexity of dynamical modularity can be mea-

TABLE II. Thermodynamic properties of flows of random walkers in compact cities at $\beta=1$ (the usual random walks). $\bar{E}$ is the internal energy, S is the entropy, F is the free energy $C_{\max}$ is the maximal values of heat capacity, and $\beta_{C_{\max}}$ is the points they are achieved.

	$\bar{E}(\beta=1)$	$S(\beta=1)$	$F(\beta=1)$	$C_{\max}$	$\beta_{C_{\max}}$
Rothenburg o.d.T.	0.946604	3.885109	2.93850	1.60468	11.1
Bielefeld downtown	0.957324	3.89042	2.93309	2.04195	9.90
Venice canals	0.95082	4.53962	3.58880	1.92003	27.4
Amsterdam canals	0.898409	3.88836	2.98996	1.3353, 1.1094	8.6, 48.75
Manhattan	0.990083	5.86713	4.87705	5.87368	13.65

sured by the information quantity. The information versus correlation profile is an individual city fingerprint.

The statistical mechanics description of random walks in compact cities provides us with a rigorous definition of the unique measure and all statistical moments which have a definite relation to the entire topological properties of complex networks. Due to a complicated topology of city streets and canals, the flows of random walkers acquire nontrivial thermodynamic characteristics which can be implemented in studies of the interactions between different city modules and of city stability with respect to occasional accessibility problems that would be of vital importance for city traffic conditions in urgency.

The obvious advantage of the method is that the information on the role of a given street or canal in the entire city network can be interpreted by the very end, and all city modules can be named street by street.

The method can be generalized for complex networks which can be described by directed or weighted graphs. For instance, one can consider a graph of a city plan in which the weights of edges are the actual lengths of streets. In the case of the weighted adjacency matrix, the spectrum of the relevant Laplace operator acquires pairs of complex-conjugated eigenvalues. Then the components of eigenvectors are also complex, and the coefficients of linear correlations should be computed for the real and imaginary parts of the spectrum separately. Other approaches could be used while studying directed graphs for which the adjacency matrix is not symmetric. For the random walks defined on the directed graphs, the probability that a random walker enters a node is not equal to the probability it leaves the node. The ensembles of random walkers introduced on the directed graphs can be described by Nelson stochastic mechanics [51,52]. The stationary configurations of random walkers in such models can be studied by means of biorthogonal decomposition [53].

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